

Prove

2. $\csc x \sin x = \sec x \cos x$
3. $\cot x \sin x = \cos x$
4. $\tan x = -\tan(-x)$
5. $\frac{\sin^2 x}{\cos x} = \sec x - \cos x$
6. $\frac{\csc x}{\cos x} = \tan x + \cot x$
7. $(\cos^2 x)(\cot^2 x + 1) = \cot^2 x$
8. $\frac{\sin \alpha \csc \alpha}{\cot \alpha} = \tan \alpha$
9. $\frac{\sin^2 u - \cos^2 u}{\sin u \cos u} = \tan u - \cot u$
10. $\frac{\sec \theta - \csc \theta}{\sec \theta \csc \theta} = \sin \theta - \cos \theta$
11. Using $\cos(x + y) = \cos x \cos y - \sin x \sin y$, show that $\cos(x + 2\pi) = \cos x$.
12. Using $\sin(x + y) = \sin x \cos y + \cos x \sin y$, show that $\sin(x + \pi) = -\sin x$.

In Problems 13 and 14, verify each identity for the indicated value.

13. $\cos 2x = 1 - 2 \sin^2 x$, $x = 30^\circ$
14. $\sin \frac{x}{2} = \pm \sqrt{\frac{1 - \cos x}{2}}$, $x = \frac{\pi}{2}$
- ☆ 15. Write $\sin 8t \sin 5t$ as a sum or difference.
- ☆ 16. Write $\sin w + \sin 5w$ as a product.

Verify each identity in Problems 17–20.

17. $\frac{1 - \cos^2 t}{\sin^3 t} = \csc t$
18. $\frac{(\cos \alpha - 1)^2}{\sin^2 \alpha} = \frac{1 - \cos \alpha}{1 + \cos \alpha}$
19. $\frac{1 - \tan^2 x}{1 - \tan^4 x} = \cos^2 x$
27. $\frac{\sin x}{1 - \cos x} = (\csc x)(1 + \cos x)$
28. $\frac{1 - \tan^2 x}{1 - \cot^2 x} = 1 - \sec^2 x$
29. $\tan(x + \pi) = \tan x$
30. $1 - (\cos \beta - \sin \beta)^2 = \sin 2\beta$
31. $\frac{\sin 2x}{\cot x} = 1 - \cos 2x$
32. $\frac{2 \tan x}{1 + \tan^2 x} = \sin 2x$
33. $2 \csc 2x = \tan x + \cot x$
34. $\csc x = \frac{\cot(x/2)}{1 + \cos x}$
35. $\frac{\sin(x - y)}{\sin(x + y)} = \frac{\tan x - \tan y}{\tan x + \tan y}$
36. $\csc 2x = \frac{\tan x + \cot x}{2}$

37. $\frac{2 - \sec^2 x}{\sec^2 x} = \cos 2x$
38. $\tan \frac{x}{2} = \frac{\sec x - 1}{\tan x}$
- ☆ 39. $\frac{\sin t + \sin 5t}{\cos t + \cos 5t} = \tan 3t$
- ☆ 40. $\frac{\sin x + \sin y}{\cos x - \cos y} = -\cot \frac{x - y}{2}$
- ☆ 41. $\frac{\cos x - \cos y}{\cos x + \cos y} = -\tan \frac{x + y}{2} \tan \frac{x - y}{2}$

Answers

$$\begin{aligned} 2. \quad \csc x \sin x &= \frac{1}{\sin x} \sin x \\ &= 1 \\ &= \frac{1}{\cos x} \cos x \\ &= \sec x \cos x \end{aligned}$$

Reciprocal identity

Algebra

Algebra

Reciprocal identity

$$\begin{aligned} 3. \quad \cot x \sin x &= \frac{\cos x}{\sin x} \sin x \\ &= \cos x \end{aligned}$$

Quotient identity

Algebra

$$\begin{aligned} 4. \quad \tan x &= \frac{\sin x}{\cos x} \\ &= \frac{-\sin(-x)}{\cos(-x)} \\ &= -\tan(-x) \end{aligned}$$

Quotient identity

Identities for negatives

Quotient identity

$$\begin{aligned} 5. \quad \frac{\sin^2 x}{\cos x} &= \frac{1 - \cos^2 x}{\cos x} \\ &= \frac{1}{\cos x} - \frac{\cos^2 x}{\cos x} \\ &= \frac{1}{\cos x} - \cos x \\ &= \sec x - \cos x \end{aligned}$$

Pythagorean identity

Algebra

Algebra

Reciprocal identity

6. $\frac{\csc x}{\cos x} = \frac{1}{\sin x \cos x}$ Reciprocal identity
 $= \frac{1}{\sin x} + \cos x$ Algebra
 $= \frac{1}{\sin x} \cdot \frac{1}{\cos x}$ Algebra
 $= \frac{1}{\sin x \cos x}$ Algebra
 $= \frac{\sin^2 x + \cos^2 x}{\sin x \cos x}$ Pythagorean identity
 $= \frac{\sin^2 x}{\sin x \cos x} + \frac{\cos^2 x}{\sin x \cos x}$ Algebra
 $= \frac{\sin x}{\cos x} + \frac{\cos x}{\sin x}$ Algebra
 $= \tan x + \cot x$ Quotient identities
7. $\cos^2 x (1 + \cot^2 x) = \cos^2 x \csc^2 x$ Pythagorean identity
 $= \cos^2 x \frac{1}{\sin^2 x}$ Reciprocal identity
 $= \frac{\cos^2 x}{\sin^2 x}$ Algebra
 $= \cot^2 x$ Quotient identity
8. $\frac{\sin \alpha \csc \alpha}{\cot \alpha} = \frac{\sin \alpha \cdot \frac{1}{\sin \alpha}}{\cot \alpha}$ Reciprocal identity
 $= \frac{1}{\cot \alpha}$ Algebra
 $= \tan \alpha$ Reciprocal identity
9. $\frac{\sin^2 u - \cos^2 u}{\sin u \cos u} = \frac{\sin^2 u}{\sin u \cos u} - \frac{\cos^2 u}{\sin u \cos u}$ Algebra
 $= \frac{\sin u}{\cos u} - \frac{\cos u}{\sin u}$ Algebra
 $= \tan u - \cot u$ Quotient identities
10. $\frac{\sec \theta - \csc \theta}{\sec \theta \csc \theta} = \frac{\sec \theta}{\sec \theta \csc \theta} - \frac{\csc \theta}{\sec \theta \csc \theta}$ Algebra
 $= \frac{1}{\csc \theta} - \frac{1}{\sec \theta}$ Algebra
 $= \sin \theta - \cos \theta$ Reciprocal identities

$$\begin{aligned}
 11. \quad \cos(x+y) &= \cos x \cos y - \sin x \sin y \\
 \cos(x+2\pi) &= \cos x \cos 2\pi - \sin x \sin 2\pi \\
 &= \cos x(1) - \sin x(0) \\
 &= \cos x
 \end{aligned}$$

$$\begin{aligned}
 12. \quad \sin(x+y) &= \sin x \cos y + \cos x \sin y \\
 \sin(x+\pi) &= \sin x \cos \pi + \cos x \sin \pi \\
 &= \sin x(-1) + \cos x(0) \\
 &= -\sin x
 \end{aligned}$$

$$\begin{aligned}
 13. \quad \cos 2x &= \cos 2(30^\circ) = \cos 60^\circ = \frac{1}{2} \\
 1 - 2 \sin^2 x &= 1 - 2 \sin^2 (30^\circ) \\
 &= 1 - 2(\sin 30^\circ)^2 \\
 &= 1 - 2\left(\frac{1}{2}\right)^2 = 1 - \frac{1}{2} \\
 &= \frac{1}{2}
 \end{aligned}$$

$$14. \quad \sin \frac{x}{2} = \sin \frac{\pi/2}{2} = \sin \frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

Since $\frac{\pi}{4}$ is in the first quadrant, the sign of the square root is chosen to be positive.

$$\sqrt{\frac{1 - \cos x}{2}} = \sqrt{\frac{1 - \cos \pi/2}{2}} = \sqrt{\frac{1 - 0}{2}} = \sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}}$$

$$\begin{aligned}
 15. \quad \sin x \sin y &= \frac{1}{2} [\cos(x-y) - \cos(x+y)] && \text{Let } x = 8t \text{ and } y = 5t \\
 \sin 8t \sin 5t &= \frac{1}{2} [\cos(8t-5t) - \cos(8t+5t)] = \frac{1}{2} (\cos 3t - \cos 13t) = \frac{1}{2} \cos 3t - \frac{1}{2} \cos 13t
 \end{aligned}$$

$$\begin{aligned}
 16. \quad \sin x + \sin y &= 2 \sin \frac{x+y}{2} \cos \frac{x-y}{2} && \text{Let } x = w \text{ and } y = 5w \\
 \sin w + \sin 5w &= 2 \sin \frac{w+5w}{2} \cos \frac{w-5w}{2} = 2 \sin 3w \cos(-2w) = 2 \sin 3w \cos 2w
 \end{aligned}$$

$$\begin{aligned}
 17. \quad \frac{1 - \cos^2 t}{\sin^3 t} &= \frac{\sin^2 t}{\sin^3 t} && \text{Pythagorean identity} \\
 &= \frac{1}{\sin t} && \text{Algebra} \\
 &= \csc t && \text{Reciprocal identity}
 \end{aligned}$$

$$\begin{aligned}
 18. \quad \frac{(\cos \alpha - 1)^2}{\sin^2 \alpha} &= \frac{(\cos \alpha - 1)^2}{1 - \cos^2 \alpha} && \text{Pythagorean identity} \\
 &= \frac{(\cos \alpha - 1)(\cos \alpha - 1)}{(1 - \cos \alpha)(1 + \cos \alpha)} && \text{Algebra} \\
 &= \frac{(-1)(\cos \alpha - 1)}{1 + \cos \alpha} && \text{Algebra} \\
 &= \frac{1 - \cos \alpha}{1 + \cos \alpha} && \text{Algebra}
 \end{aligned}$$

$$\text{Key algebraic steps: } \frac{(b-1)^2}{1-b^2} = \frac{(b-1)(b-1)}{(1-b)(1+b)} = \frac{(-1)(b-1)}{1+b} = \frac{1-b}{1+b}$$

$$\begin{aligned}
 19. \quad \frac{1 - \tan^2 x}{1 - \tan^4 x} &= \frac{1 - \tan^2 x}{(1)^2 - (\tan^2 x)^2} && \text{Algebra} \\
 &= \frac{1 - \tan^2 x}{(1 - \tan^2 x)(1 + \tan^2 x)} && \text{Algebra} \\
 &= \frac{1}{1 + \tan^2 x} && \text{Algebra} \\
 &= \frac{1}{\sec^2 x} && \text{Pythagorean identity} \\
 &= \left(\frac{1}{\sec x}\right)^2 && \text{Algebra} \\
 &= \cos^2 x && \text{Reciprocal identity}
 \end{aligned}$$

$$\text{Key algebraic steps: } \frac{1-c^2}{1-c^4} = \frac{1-c^2}{(1)^2 - (c^2)^2} = \frac{1-c^2}{(1-c^2)(1+c^2)} = \frac{1}{1+c^2}$$

$$\begin{aligned}
 20. \quad \cot^2 x \cos^2 x &= (\csc^2 x - 1) \cos^2 x && \text{Pythagorean identity} \\
 &= \csc^2 x \cos^2 x - \cos^2 x && \text{Algebra} \\
 &= \left(\frac{1}{\sin x}\right)^2 \cos^2 x - \cos^2 x && \text{Reciprocal identity} \\
 &= \left(\frac{\cos x}{\sin x}\right)^2 - \cos^2 x && \text{Algebra} \\
 &= \cot^2 x - \cos^2 x && \text{Quotient identity}
 \end{aligned}$$

$$\begin{aligned}
27. \quad \frac{\sin x}{1 - \cos x} &= \frac{\sin x}{1 - \cos x} \frac{1 + \cos x}{1 + \cos x} \\
&= \frac{\sin x (1 + \cos x)}{1 - \cos^2 x} \\
&= \frac{\sin x (1 + \cos x)}{\sin^2 x} \\
&= \frac{\sin x}{\sin^2 x} (1 + \cos x) \\
&= \frac{1}{\sin x} (1 + \cos x) \\
&= \csc x (1 + \cos x)
\end{aligned}$$

Algebra

Algebra

Pythagorean identity

Algebra

Algebra

Reciprocal identity

$$\begin{aligned}
28. \quad \frac{1 - \tan^2 x}{1 - \cot^2 x} &= \frac{1 - \tan^2 x}{1 - \frac{1}{\tan^2 x}} && \text{Reciprocal identity} \\
&= \frac{\tan^2 x (1 - \tan^2 x)}{\tan^2 x \left(1 - \frac{1}{\tan^2 x}\right)} && \text{Algebra} \\
&= \frac{\tan^2 x (1 - \tan^2 x)}{\tan^2 x - 1} && \text{Algebra} \\
&= \frac{-\tan^2 x (\tan^2 x - 1)}{\tan^2 x - 1} && \text{Algebra} \\
&= -\tan^2 x && \text{Algebra} \\
&= -(\sec^2 x - 1) && \text{Pythagorean identity} \\
&= 1 - \sec^2 x && \text{Algebra}
\end{aligned}$$

Key algebraic steps: $\frac{1 - a^2}{1 - \frac{1}{a^2}} = \frac{a^2(1 - a^2)}{a^2\left(1 - \frac{1}{a^2}\right)} = \frac{a^2(1 - a^2)}{a^2 - 1} = \frac{-a^2(a^2 - 1)}{a^2 - 1} = -a^2$

$$\begin{aligned}
29. \quad \tan(x + \pi) &= \frac{\tan x + \tan \pi}{1 - \tan x \tan \pi} && \text{Sum identity} \\
&= \frac{\tan x + 0}{1 - \tan x \cdot 0} && \text{Known values} \\
&= \tan x && \text{Algebra}
\end{aligned}$$

$$\begin{aligned}
30. \quad 1 - (\cos \beta - \sin \beta)^2 &= 1 - (\cos^2 \beta - 2 \sin \beta \cos \beta + \sin^2 \beta) && \text{Algebra} \\
&= 1 - \cos^2 \beta + \sin^2 \beta - 2 \sin \beta \cos \beta && \text{Algebra} \\
&= 1 - (1 - 2 \sin \beta \cos \beta) && \text{Pythagorean identity} \\
&= 1 - 1 + 2 \sin \beta \cos \beta && \text{Algebra} \\
&= 2 \sin \beta \cos \beta && \text{Algebra} \\
&= \sin 2\beta && \text{Double-angle identity}
\end{aligned}$$

$$\begin{aligned}
31. \quad \frac{\sin 2x}{\cot x} &= \frac{2 \sin x \cos x}{\cot x} && \text{Double-angle identity} \\
&= \frac{2 \sin x \cos x}{\frac{\cos x}{\sin x}} && \text{Quotient identity} \\
&= 2 \sin x \cos x + \frac{\cos x}{\sin x} && \text{Algebra} \\
&= 2 \sin x \cos x \cdot \frac{\sin x}{\cos x} && \text{Algebra} \\
&= 2 \sin^2 x && \text{Algebra} \\
&= 2 \sin^2 x - 1 + 1 && \text{Algebra} \\
&= 1 + (2 \sin^2 x - 1) && \text{Algebra} \\
&= 1 - (1 - 2 \sin^2 x) && \text{Algebra} \\
&= 1 - \cos 2x && \text{Double-angle identity}
\end{aligned}$$

32.	$\frac{2 \tan x}{1 + \tan^2 x} = \frac{2 \frac{\sin x}{\cos x}}{1 + \frac{\sin^2 x}{\cos^2 x}}$ $= \frac{\cos^2 x \cdot 2 \frac{\sin x}{\cos x}}{\cos^2 x \cdot 1 + \cos^2 x \cdot \frac{\sin^2 x}{\cos^2 x}}$ $= \frac{2 \sin x \cos x}{\cos^2 x + \sin^2 x}$ $= \frac{2 \sin x \cos x}{1}$ $= 2 \sin x \cos x$ $= \sin 2x$	<p>Quotient identity</p> <p>Algebra</p> <p>Algebra</p> <p>Pythagorean identity</p> <p>Algebra</p> <p>Double-angle identity</p>
33.	$2 \csc 2x = \frac{2}{\sin 2x}$ $= \frac{2}{2 \sin x \cos x}$ $= \frac{1}{\sin x \cos x}$ $= \frac{\sin^2 x + \cos^2 x}{\sin x \cos x}$ $= \frac{\sin^2 x}{\sin x \cos x} + \frac{\cos^2 x}{\sin x \cos x}$ $= \frac{\sin x}{\cos x} + \frac{\cos x}{\sin x}$ $= \tan x + \cot x$	<p>Reciprocal identity</p> <p>Double-angle identity</p> <p>Algebra</p> <p>Pythagorean identity</p> <p>Algebra</p> <p>Algebra</p> <p>Quotient identities</p>
34.	$\frac{\cot \frac{x}{2}}{1 + \cos x} = \cot \frac{x}{2} \cdot \frac{1}{1 + \cos x}$ $= \frac{1}{\tan \frac{x}{2}} \cdot \frac{1}{1 + \cos x}$ $= \frac{1}{\frac{\sin x}{1 + \cos x}} \cdot \frac{1}{1 + \cos x}$ $= \frac{1 + \cos x}{\sin x} \cdot \frac{1}{1 + \cos x}$ $= \frac{1}{\sin x}$ $= \csc x$	<p>Algebra</p> <p>Reciprocal identity</p> <p>Half-angle identity</p> <p>Algebra</p> <p>Algebra</p> <p>Reciprocal identity</p>

35. $\frac{\sin(x-y)}{\sin(x+y)} = \frac{\sin x \cos y - \cos x \sin y}{\sin x \cos y + \cos x \sin y}$ Sum and difference identities
- $$= \frac{\frac{\sin x \cos y}{\cos x \cos y} - \frac{\cos x \sin y}{\cos x \cos y}}{\frac{\sin x \cos y}{\cos x \cos y} + \frac{\cos x \sin y}{\cos x \cos y}}$$
- Algebra
- $= \frac{\frac{\sin x}{\cos x} - \frac{\sin y}{\cos y}}{\frac{\sin x}{\cos x} + \frac{\sin y}{\cos y}}$ Algebra
- $$= \frac{\tan x - \tan y}{\tan x + \tan y}$$
- Quotient identity
36. $\csc 2x = \frac{1}{\sin 2x}$ Reciprocal identity
- $$= \frac{1}{2 \sin x \cos x}$$
- Double-angle identity
- $= \frac{\sin^2 x + \cos^2 x}{2 \sin x \cos x}$ Pythagorean identity
- $$= \frac{\sin^2 x}{2 \sin x \cos x} + \frac{\cos^2 x}{2 \sin x \cos x}$$
- Algebra
- $$= \frac{\sin x}{2 \cos x} + \frac{\cos x}{2 \sin x}$$
- Algebra
- $$= \frac{1}{2} \frac{\sin x}{\cos x} + \frac{1}{2} \frac{\cos x}{\sin x}$$
- Algebra
- $$= \frac{1}{2} \tan x + \frac{1}{2} \cot x$$
- Quotient identities
- $$= \frac{\tan x + \cot x}{2}$$
- Algebra
37. $\frac{2 - \sec^2 x}{\sec^2 x} = \frac{1 - \frac{1}{\cos^2 x}}{\frac{1}{\cos^2 x}}$ Reciprocal identity
- $$= \frac{\cos^2 x \cdot 2 - \cos^2 x \cdot \frac{1}{\cos^2 x}}{\cos^2 x \cdot \frac{1}{\cos^2 x}}$$
- Algebra
- $$= \frac{2 \cos^2 x - 1}{1}$$
- Algebra
- $$= 2 \cos^2 x - 1$$
- Algebra
- $$= \cos 2x$$
- Double-angle identity

$$\begin{aligned}
 38. \quad \tan \frac{x}{2} &= \frac{1 - \cos x}{\sin x} && \text{Half-angle identity} \\
 &= \frac{1 - \cos x}{\cos x - \cos x} && \text{Algebra} \\
 &= \frac{\sin x}{\cos x} && \text{Reciprocal identity} \\
 &= \frac{\sec x - \cos x}{\cos x} && \text{Algebra} \\
 &= \frac{\sec x - 1}{\sin x} && \text{Quotient identity} \\
 &= \frac{\sec x - 1}{\tan x}
 \end{aligned}$$

$$\begin{aligned}
 39. \quad \frac{\sin t + \sin 5t}{\cos t + \cos 5t} &= \frac{2 \sin \frac{t+5t}{2} \cos \frac{t-5t}{2}}{2 \cos \frac{t+5t}{2} \cos \frac{t-5t}{2}} && \text{Sum-product identities} \\
 &= \frac{2 \sin 3t \cos(-2t)}{2 \cos 3t \cos(-2t)} && \text{Algebra} \\
 &= \frac{\sin 3t}{\cos 3t} && \text{Algebra} \\
 &= \tan 3t && \text{Quotient identity}
 \end{aligned}$$

$$\begin{aligned}
 40. \quad \frac{\sin x + \sin y}{\cos x - \cos y} &= \frac{2 \sin \frac{x+y}{2} \cos \frac{x-y}{2}}{-2 \sin \frac{x+y}{2} \sin \frac{x-y}{2}} && \text{Sum-product identities} \\
 &= -\frac{\cos \frac{x-y}{2}}{\sin \frac{x-y}{2}} && \text{Algebra} \\
 &= -\cot \frac{x-y}{2} && \text{Quotient identity}
 \end{aligned}$$

$$\begin{aligned}
 41. \quad \frac{\cos x - \cos y}{\cos x + \cos y} &= \frac{-2 \sin \frac{x+y}{2} \sin \frac{x-y}{2}}{2 \cos \frac{x+y}{2} \cos \frac{x-y}{2}} && \text{Sum-product identities} \\
 &= -\frac{\sin \frac{x+y}{2} \sin \frac{x-y}{2}}{\cos \frac{x+y}{2} \cos \frac{x-y}{2}} && \text{Algebra} \\
 &= -\tan \frac{x+y}{2} \tan \frac{x-y}{2} && \text{Quotient identity}
 \end{aligned}$$